The Isoperimetric Inequality: A History of the Problem, Proofs and Applications

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1 Introduction

The isoperimetric problem can be stated two ways: (1) maximize the area enclosed by a closed curve of length L; (2) minimize the length of a curve bounding a region of area A. This task leads to the isoperimetric inequality:

$$L^2 \geqslant 4\pi A \tag{1}$$

where L is the length of the closed curve and A is the area of the bounded region. For Euclidean geometry the result is a circle. It should be noted that this fact was known in Ancient Greece. Jakob Steiner brought attention to the problem in the modern world by showing in 1838 that a circle must be the answer if a solution exists. His solution used the method that if part of a curve was concave it is possible to flip the curve to make it convex giving more area for the same length.

The problem can also be generalized to higher dimensions, but it should be noted that many of the 2D methods to prove the inequality no longer hold. In 3D one maximizes the volume with respect to the surface area; in Euclidean geometry we get a sphere. For even higher dimensions the hyperdimensional volume is maximized with respect to the hyperdimensional surface area.

The inequality implies that a surface with constant mean curvature encloses the greatest volume in relation to the surface area. While such a surface seems like it should be a sphere it took till 1900 to get a basic proof which was improved in the 1950's. Soap bubbles have constant mean curvature related to the pressure across the soap film. As has been observed, a soap bubble with a constant pressure difference across the entirity of its surface area takes the form of a sphere. See [7] for a discussion on the geometry of soap bubbles as well as the physical processes dictating their structure.

1.1 Zenodorus

Zenodorus was an ancient Greek mathematician from around 200 B.C to 120 B.C. His most important work was *On isometric figures*, which has sadly been lost, in which Zenodorus examined figures with equal perimeters but different shapes. Parts of his work survived through references by other mathematicians such as Pappus and Theon of Alexandria.

1.1.1 Proofs and Theories

Zenodorus managed to prove many important statements which suggested the isoperimetric problem, but the mathematics of the time was not advanced enough to prove the problem itself. Despite these limitations Zenodorus still proved:

(1) The regular polygon with most angles had the greatest area.

(2) The circle has greater area than any regular polygon of equal perimeter.

(3) The equilateral and equiangular (in other words, regular) polygon has greatest area of any polygon with same perimeter and number of sides.

He also theorized that sphere has the greatest volume of any solid surface with the same surface area which is the answer to the 3D isoperimetric problem.

1.2 Two Equivalent Statements

For closed curves in a plane:

(A) Of all such curves with fixed perimeter, the curve that forms a circle encloses the greatest area.

(B) Of all such curves enclosing a fixed area, the curve that forms a circle has the shortest perimeter.

It can be shown that if we assume A, then we can show B and vice versa.[1]

2 A Brief History

The isoperimetric problem has been known since the time of the early Greeks. It was one of the first optimization problems, of particular importance for land measurement. Thucydides and other historians as well as geometers around 100 B.C. measured the size of a city by the time it takes to circumnavigate it. Thucydides defined the size of Sicily as the time it took to sail around it. Proclus, a classical mathematician, mocked geometers for "measuring the size of a city from the lengths of its walls."

It was not obvious to the common person of antiquity that two shapes of equal perimeter could have different areas. Interestingly, some people cheated others out of land by taking advantage of this misconception. Even more amusing, these cheaters were considered generous which goes to show just how unintuitive the concept of shapes with the same perimeter having different areas was to the ancient Greeks.

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[1] Nahin, 55

2.1 Dido's Problem

Dido's problem is one of the most famous math problems of antiquity. In literatire it was first noted in Virgil's *Aeneid.* The story goes that queen Dido was chased away from her home in Phoenicia by her brother. She went to Africa and made an agreement with the natives to purchase a piece of land which she could enclose with a bull's hide.

> *"The Kingdom you see is Carthage, the Tyrians, the town of Agenor;*
> *But the country around is Libya, no folk to meet in war.*
> *Dido, who left the city of Tyre to escape her brother,*
> *Rules here–a long a labyrinthine tale of wrong*
> *Is hers, but I will touch on its salient points in order....Dido, in great disquiet, organised her friends for escape.*
> *They met together, all those who harshly hated the tyrant*
> *Or keenly feared him: they seized some ships which chanced to be ready...*
> *They came to this spot, where to-day you can behold the mighty*
> *Battlements and the rising citadel of New Carthage,*
> *And purchased a site, which was named 'Bull's Hide' after the bargain*
> *By which they should get as much land as they could enclose with a bull's hide."*

The *Aeneid* was written between 29 and 19 B.C. so it is clear that Dido's problem, and hence the isoperimetric problem, has been known since at least that time period.

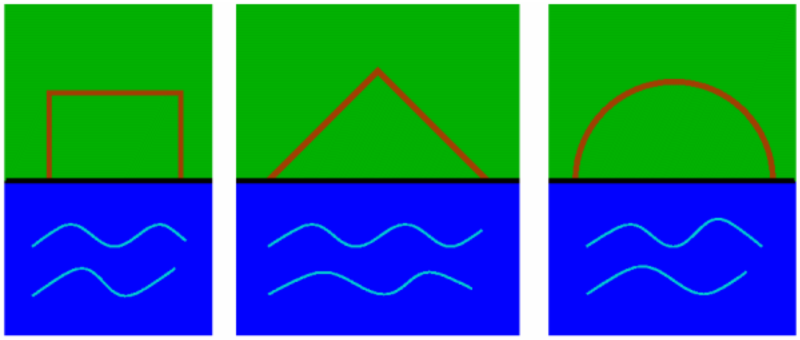

Figure 1: Representations of areas bounded by common shapes of the same perimeter. The semicircle (answer to Dido's problem) contains the greatest area.

Historia Regum Britanniae (History of the Kings of England) by Geoffrey of Monmouth written in the 12th century A.D. also makes note of this problem where a German Duke Hengist offered military service to King Vortigern in exchange for the land he could enclose with a bull's hide. With this land he established the castle of Kaercorrei.

2.2 Historic Proofs

2.2.1 Steiner

It was not until the mid 19th century that a proof was found for the classical isoperimetric problem. Jakob Steiner, a Swiss mathematician led the way with his four-hinge method in 1838. As noted above, Steiner did not succeed in a full proof as he was not above to prove the existence of a solution. Carathedory completed Steiner's proof.

Steiner's proof relies on two facts that are readily accessible at the high school level.

(1) Any inscibed triangle in a circle with a diameter as its hypotenuse has a right angle between its legs.

(2) Right triangles have the largest area of any triangle with the same legs.

This last point is obvious, given legs X and Y with angle θ between them:

$$A = \frac{1}{2}XY\sin(\theta)$$

Which is maximized for $\sin(\theta) = 1$.

Suppose we have the following initial curve.

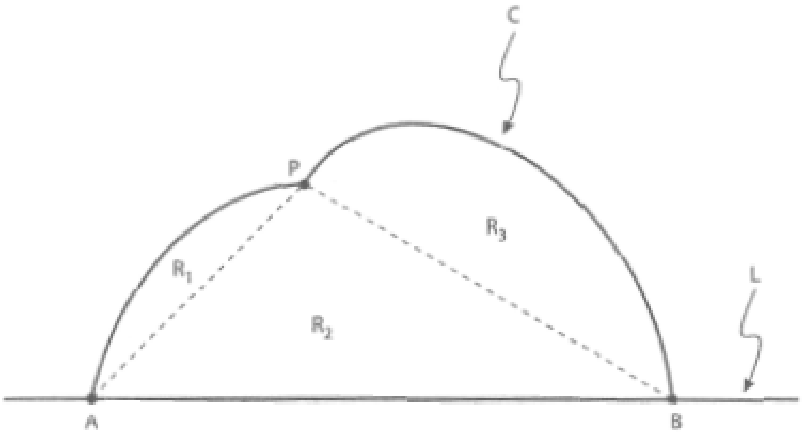

Figure 2: Initial Curve [Nahin, 57]

Steiner symmetrization works by choosing a point P and forming a triangle between that point and the endpoints on the line of the semicircle. As can be seen above, AP, PB, AB form a triangle. Let us then move the points A and B on the line such that $< APB$ is a right angle. By doing so we keep the lengths AP and BP the same. Thus from (2) above, out new triangle is bigger than our old one.

If we divide up the inital curve into the area inside the triangle, R_2, and then the area in the other parts of the curve, R_1 and R_3, the area bounded by the curve after this symmetrization is larger. R_1 and R_3 stay the same but $R_2' > R_2$ so the sum of the areas is greater.

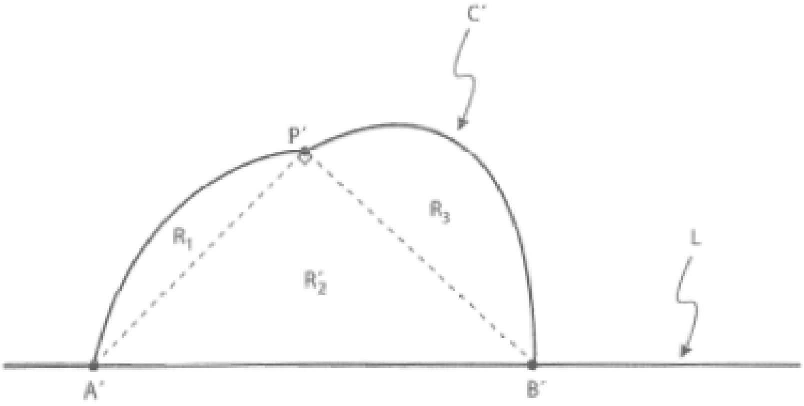

Figure 3: The result of one iteration. [Nahin, 58]

As the number of iterations increases the curve tends toward a circle which would be the result of an infinite number of iterations. In each iteration the area bounded by the curve can never decrease. Thus we conclude that the circle has the greatest internal area for its measured perimeter.

As noted above, this proof only shows the uniqueness of the circle as the solution. It does not show existence, as noted by the German mathematician Peter Dirichlet . That is, it remains to be shown that there is a curve that maximizes area with respect to the perimeter.

2.2.2 The First Rigorous Proof

While Steiner had made a significant contribution to the understanding of the isoperimetric problem, his proof was not complete. The first complete proof was developed by Weierstrass using variational calculus. This method was also used to prove higher dimensional versions of the isoperimetric problem. Schwartz used Steiner symmetrization to show the 3D case in 1884.

For higher dimesions the problem can be stated: Maximize the hypervolume with respect to the hyper-surface area. Below is the outline of the 2D proof using variational calculus.

Proof:

Define the plane curve C as:

$$r(t) = (x(t), y(t))$$

We can then define the area and length:

$$A = \frac{1}{2}\int_o^T \left(y(t)\frac{dx}{dt} - x(t)\frac{dy}{dt} \right) dt = \int_o^T F(t)dt$$

$$L = \int ds = \int_o^T \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_o^T G(t)dt$$

We then define

$$H(t) = F(t) + \lambda G(t)$$

where λ is the Lagrange multiplier.

We get two Euler-Lagrange equations:

$$\frac{\partial H}{\partial x} - \frac{d}{dt}\left(\frac{\partial H}{\partial x_t}\right) = 0$$

$$\frac{\partial H}{\partial y} - \frac{d}{dt}\left(\frac{\partial H}{\partial y_t}\right) = 0$$

Where x_t, y_t is the time derivative of x and y respectively. Solving this system of equations gives the solution:

$$(y(t) - C_1)^2 + (x(t) - C_2)^2 = \lambda^2$$

For which C_1 and C_2 are constants of integration. The form of this equation is that of a circle with radius λ.

Generalizing this method to higher dimensions, the $r(t)$ equation gains extra terms which affects the $F(t)$ and $G(t)$ from above. But once they are found the method is the same. This method produces N Euler-Langrange equations which N is the dimensionality of the curve being considered.

2.2.3 Other Methods of Proof

There are many methods by which the isoperimetric problem has been proved. The methods for solving the 2D version are diverse. A few methods for the 2D case which have met success are Steiner's four-hinge method, symmetrization, variational calculus, approximating polyhedra, series of trigonometric terms, integral geometryand equidistants.

For the 3D and ND generalizations the following methods have been used: variational calculus, Brunn-Minkowski inequality and convexity. The number of methods for the higher dimension generalization is significantly fewer than that for the 2D case.

3 Many Applications

The applications that the isoperimetric problem has found use in are diverse. See [11] for a discussion on several implications of the isoperimetric inequality. I will use the rest of the paper to discuss an ancient analysis which uses the isoperimetric inequality.

3.1 Pappus' Bees

Pappus was one of the latest classical mathematicians following Zenodorus by 300 years putting Pappus somewhere between 100 A.D. and 200 A.D. It is through Pappus' work as well as references from other classical mathematicians that we are aware of Zenodorus' work, as well as other mathematical works, which was unforunately lost to history. Pappus is most famous for his analysis of the hexagonal structure of the honeycomb structure in a bee hive and its relationship to the isoperimetric problem titled *On the Sagacity of Bees*.

The surface of a plane can only be tiled by three regular polygon shapes. The equilateral triangle, square and hexagon. This fact has to do with the fact that the external angle is a whole number multiple of the internal angle for only these three shapes.

The total internal angle for a polygon with n sides is $180(n-2)$. We consider only regular polygons using the result of Zenodorus' proof of (3) above.

n	Internal angle sum	External angle	Internal angle	$\frac{External\ angle}{Internal\ angle}$
3*	180	300	60	5*
4*	360	270	90	3*
5	540	252	108	2.333
6*	720	240	120	2*
7	900	231.429	128.571	1.8
8	1080	225	135	1.667

Table 1: Table of Internal and corresponding External angles for regular polygons of n sides. The starred entries are those which have an integer multiple of the internal angle for the external angle. This means that these polygons can be tiled on a plane.

Thus we get the following possible tilings:

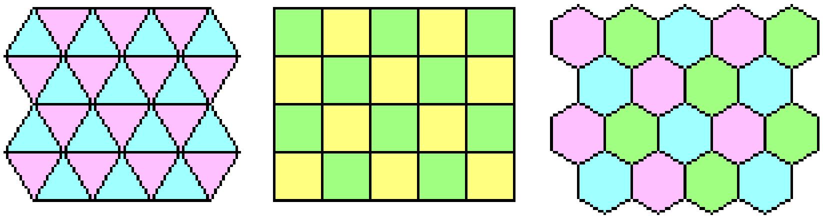

Figure 4: The three only possible tilings on the surface of a plane using regular polygons.

The regular triangle, square and hexagon all tile the plane without gaps. However Zenodorus showed in (1) that the regular polygon with the most angles has the most area. From these three regular polygons, the hexagon has the most angles, and thus it has the greatest area for its perimeter. We conclude that the hexagon is the most efficient shape for tiling the surface of a plane.

What is interesting, both in modern times and what Pappus made note of, is that bees figured this optimization out. In this case nature has an example of this most efficient tiling which takes into account the fact that there are only three regular polygons which can tile a plane as well as Zenodorus' observation that the regular polygon with the most angles bounds the greatest area.

4 References

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